1.

1. Substituting $n = 0$ gives us:

$$I_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx = [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2$$

Substituing $n = 1$ gives:

$$\begin{aligned}
\frac{I_1}{q^2} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right) \cos x dx \\
&= \left[\left(\frac{\pi^2}{4} - x^2\right) \sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-2x)(\sin x) dx && \text{(Integrating by parts)} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2x \sin x dx \\
&= [2x(-\cos x)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-\cos x) dx && \text{(Integrating by parts)} \\
&= 4 \\
\implies I_1 &= 4q^2
\end{aligned}$$

$$I_n = \frac{q^{2n}}{n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^n \cos x dx$$

$$= \frac{q^{2n}}{n!} \left\{ \left[\left(\frac{\pi^2}{4} - x^2\right)^n (\sin x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} n \left(\frac{\pi^2}{4} - x^2\right)^{n-1} (-2x)(\sin x) dx \right\}$$

$$= \frac{q^{2n}}{n!} \left\{ 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \left(\frac{\pi^2}{4} - x^2\right)^{n-1} \sin x dx \right\}$$

$$= \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \left(\frac{\pi^2}{4} - x^2\right)^{n-1} \sin x dx$$

$$= \frac{2q^{2n}}{(n-1)!} \left\{ \left[x \left(\frac{\pi^2}{4} - x^2\right)(-\cos x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(\frac{\pi^2}{4} - x^2\right)^{n-1} + (n-1)x \left(\frac{\pi^2}{4} - x^2\right)^{n-2} (-2x)\right] (-\cos x) dx \right\}$$

$$= \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-1} \cos x dx - \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (n-1)(2x^2) \left(\frac{\pi^2}{4} - x^2\right)^{n-2} dx$$

$$= \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-1} \cos x dx - \frac{4q^{2n}}{(n-2)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \left(\frac{\pi^2}{4} - x^2\right)^{n-2} \cos x dx \qquad \blacksquare$$

3. Observe:

$$\frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-1} \cos x dx = 2q^2 \cdot \frac{q^{2(n-1)}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-1} \cos x dx$$

$$= 2q^2 I_{n-1}$$

Also,

$$\frac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\left(\frac{\pi^2}{4}-x^2\right)^{n-2}\cos x dx$$
$$=\frac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{\pi^2}{4}-\left(\frac{\pi^2}{4}-x^2\right)\right]\left(\frac{\pi^2}{4}-x^2\right)^{n-2}\cos x dx$$
$$=\frac{4q^{2n}}{(n-2)!}\cdot\frac{\pi^2}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^{n-2}\cos x dx-\frac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^{n-1}\cos x dx$$
$$=q^4\pi^2\cdot\frac{q^{2(n-2)}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^{n-2}\cos x dx-4q^2(n-1)\frac{q^{2(n-1)}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^{n-1}\cos x dx$$
$$=q^4\pi^2 I_{n-2}-4q^2(n-1)I_{n-1}$$
$$=p^2q^2 I_{n-2}-4q^2(n-1)I_{n-1} \qquad \left(\because \pi=\frac{p}{q}\right)$$

Thus,

$$\begin{aligned}I_n&=2q^2I_{n-1}-\left[p^2q^2I_{n-2}-4q^2(n-1)I_{n-1}\right]\\&=(2q^2+4q^2n-4q^2)I_{n-1}-p^2q^2I_{n-2}\\&=(4n-2)q^2I_{n-1}-p^2q^2I_{n-2}\end{aligned}$$ ■

It is clear that I_0 and I_1 are integers. Assume that I_n is an integer for $n \le k$ where $k \ge 2$.
By 3., we have that $I_{k+1}=(4k+2)q^2I_k-p^2q^2I_{k-1}$

By induction hypothesis, I_k and I_{k-1} are integers. As $(4k+2)q^2$ and p^2q^2 are integers as well, we have it that I_{k+1} is an integer.
Thus, I_n is an integer for $n=0,1,2,\cdots$ ■

For $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we have that $\cos x \ge 0$. Also, $\frac{\pi^2}{4}\ge x^2$.
Thus, $\left(\frac{\pi^2}{4}-x^2\right)^n\cos x\ge 0$ for $n=0,1,\cdots$
It is clear that the integrand is not identically zero. As it's strictly positive in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, giving us that $I_n\neq 0$.
$\therefore I_n>0$.

Now,

$$\begin{aligned}I_n&=\frac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^n\cos x dx\\&<\frac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^n dx\\&=\frac{2q^{2n}}{n!}\int_0^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-x^2\right)^n dx && \left(f(x)=f(-x)\implies\int_{-a}^a f(x)dx=2\int_0^a f(x)dx\right)\\&=\frac{2q^{2n}}{n!}\int_0^{\frac{\pi}{2}}\left(\frac{\pi^2}{4}-\left[\frac{\pi}{2}-x\right]^2\right)^n dx && \left(\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)\\&=\frac{2q^{2n}}{n!}\int_0^{\frac{\pi}{2}}(\pi x-x^2)^n dx\\&=\frac{2q^{2n}}{n!}\int_0^{\frac{\pi}{2}}x^n(\pi-x)^n dx\end{aligned}$$

Substitute $x=\pi t$

Substitute $x = \pi t$

$$= \frac{2q^{2n}}{n!} \int_0^{\frac{1}{2}} \pi^{2n+1} t^n (1-t)^n dt$$

$$= \pi \cdot \frac{2q^{2n}\pi^{2n}}{n!} \int_0^{\frac{1}{2}} (t-t^2)^n dt$$

$$\leq \pi \cdot \frac{2p^{2n}}{n!} \int_0^{\frac{1}{2}} \left(\frac{1}{4}\right)^n dt$$

$$= \frac{p}{q} \cdot \cancel{2} \cdot p^{2n} \cdot \frac{1}{n!} \cdot \frac{1}{2^{2n}} \cdot \frac{1}{\cancel{2}}$$

$$= \frac{p}{q} \left(\frac{p}{2}\right)^{2n} \frac{1}{n!}$$

$$\implies I_n < \frac{p}{q} \left(\frac{p}{2}\right)^{2n} \frac{1}{n!}$$

6. Let κ denote $\frac{p^2}{4}$.

$$\therefore \frac{p}{q} \left(\frac{p}{2}\right)^{2n} \frac{1}{n!} = \pi \frac{\kappa^n}{n!}$$

Let $n_0 := \lfloor \kappa \rfloor + 1$

For $n \geq n_0$:

$$\pi \frac{\kappa^n}{n!} = \pi \frac{\kappa^{n_0}}{n_0!} \cdot \frac{\kappa}{n_0+1} \cdot \frac{\kappa}{n_0+2} \cdots \frac{\kappa}{n}$$

$$\leq \pi \frac{\kappa^{n_0}}{n_0!} \cdot 1 \cdot 1 \cdots \frac{\kappa}{n} \qquad (k < n_0)$$

$$= \pi \frac{\kappa^{n_0+1}}{n_0!} \cdot \frac{1}{n}$$

Note that $\pi \frac{\kappa^{n_0+1}}{n_0!}$ is a fixed constant. Call it χ.

$$\therefore \frac{p}{q} \left(\frac{p}{2}\right)^{2n} \frac{1}{n!} \leq \frac{\chi}{n}$$

The right hand side tends to 0 as $n \to \infty$.

$\therefore \frac{\chi}{n} < 1$, for sufficiently large n.

$\therefore \frac{p}{q} \left(\frac{p}{2}\right)^{2n} \frac{1}{n!} < 1$, for sufficiently large n. ■

As we had proven in 5., $0 < I_n < \frac{p}{q} \left(\frac{p}{2}\right)^{2n} \frac{1}{n!}$.

By above, we have that there exists a natural number N such that $\frac{p}{q} \left(\frac{p}{2}\right)^{2N} \frac{1}{N!} < 1$

Thus, we have it that $0 < I_N < 1$.

This is a contradiction as we had proven that I_n was an integer for all natural n.

The only assumption we had made was that π can be written as a ratio of positive integers. As π is positive, if π were rational, it could have been written as a ratio of positive integers.

Thus, we have it that π must be irrational. ■

2.

Let $x_1, x_2, \cdots, x_n$ be the solutions of the above system of equations. Then, by hypothesis we have that:

$$\frac{x_1}{t-1^2} + \frac{x_2}{t-3^2} + \cdots + \frac{x_n}{t-(2n-1)^2} = 1$$

is an equation in t which has solutions $t = 2^2, 4^2, \cdots, 4n^2$. (Not claiming that these are the only solutions.)
Multiplying with the denominators:

(Note that this does not change the set of solutions as all non-zero.)

$$x_1(t-3^2)(t-5^2)\cdots(t-(2n-1)^2) + x_2(t-1^2)(t-5^2)\cdots(t-(2n-1)^2) + \cdots + x_n(t-1^2)(t-3^2)\cdots(t-(2n-3)^2) = (t-1^2)(t-3^2)\cdots(t-(2n-1)^2)$$

$$\implies t^{n-1}(x_1 + x_2 + \cdots + x_n) + O(t^{n-2}) = t^n - t^{n-1}(1^2 + 3^2 + \cdots + (2n-1)^2) + O(t^{n-2})$$
$$\implies t^n - t^{n-1}(1^2 + 3^2 + \cdots + (2n-1)^2 + S_n) + O(t^{n-2}) = 0$$

Where $O(t^{n-2})$ denotes that the expression is a polynomial in t with degree at most $n-2$.
As the set of solutions has not been changed, we still have that $t = 2^2, 4^2, \cdots, 4n^2$ are solutions. Now, note that the above is a polynomial of degree n. As we already have n distinct solutions, they must be the only solutions.
Using Vieta's formula, we have that the sum of roots is given by: $1^2+3^2+\cdots+(2n-1)^2+S_n = \sum_{k=1}^{n}(2k-1)^2+S_n$

$$\therefore \sum_{k=1}^{n}(2k-1)^2 + S_n = \sum_{k=1}^{n}(2k)^2$$
$$\implies S_n = \sum_{k=1}^{n}\left[(2k)^2 - (2k-1)^2\right] = \sum_{k=1}^{n}[4k-1]$$
$$\implies S_n = 2n(n+1) - n$$
$$\implies S_n = n(2n+1)$$

Now, we want S_n, that is, $n(2n+1)$ to be a perfect square.
Note that one of the ways that is possible is if n and $2n+1$ are both perfect squares. (Not claiming that this is the only way.)
Thus, we would have $n = p^2$ and $2n+1 = q^2$ for some $p, q \in \mathbb{Z}$.
Thus, giving us: $q^2 - 2p^2 = 1$.
As the above equation has infinitely many solutions, we will have infinitely many distinct values of n such that S_n is a perfect square. ■

3.

Answer: $\boxed{49.}$ Break all possible values of n into the four cases: $n = 2$, $n = 4$, $n > 4$ and n odd. By Fermat's theorem, no solutions exist for the $n = 4$ case because we may write $y^4 + \left(2^{25}\right)^4 = x^4$.

We show that for n odd, no solutions exist to the more general equation $x^n - y^n = 2^k$ where k is a positive integer. Assume otherwise for contradiction's sake, and suppose on the grounds of well ordering that k is the least exponent for which a solution exists. Clearly x and y must both be even or both odd. If both are odd, we have $(x-y)(x^{n-1} + ... + y^{n-1})$. The right factor of this expression contains an odd number of odd terms whose sum is an odd number greater than 1, impossible. Similarly if x and y are even, write $x = 2u$ and $y = 2v$. The equation becomes $u^n - v^n = 2^{k-n}$. If $k-n$ is greater than 0, then our choice k could not have been minimal. Otherwise, $k-n = 0$, so that two consecutive positive integers are perfect nth powers, which is also absurd.

For the case that n is even and greater than 4, consider the same generalization and hypotheses. Writing $n = 2m$, we find $(x^m - y^m)(x^m + y^m) = 2^k$. Then $x^m - y^m = 2^a < 2^k$. By our previous work, we see that m cannot be an odd integer greater than 1. But then m must also be even, contrary to the minimality of k.

Finally, for $n = 2$ we get $x^2 - y^2 = 2^{100}$. Factoring the left hand side gives $x - y = 2^a$ and $x + y = 2^b$, where implicit is $a < b$. Solving, we get $x = 2^{b-1} + 2^{a-1}$ and $y = 2^{b-1} - 2^{a-1}$, for a total of 49 solutions. Namely, those corresponding to $(a, b) = (1, 99), (2, 98), \cdots, (49, 51)$.

4.

Let $x_1, x_2, \cdots, x_{2k+1}$ be $2k+1$ variables which are randomly chosen with uniform distribution in $(0,1)$.($k \in \mathbb{N} \cup \{0\}$)

$$N := \sum_{i=1}^{2k+1} \left\lfloor \frac{1}{x_i} \right\rfloor$$

Let $P(k)$ denote the probability that N is odd. Hence, evaluate:

$$\lim_{n\to\infty} \sum_{k=0}^{n} (2P(k)-1)$$

First, let's evaluate the probability that $\left\lfloor \frac{1}{x_i} \right\rfloor$ is odd.
It is easy to observe that $\left\lfloor \frac{1}{x_i} \right\rfloor$ is odd iff $x_i \in (\frac{1}{2}, 1) \bigcup (\frac{1}{4}, \frac{1}{3}) \bigcup (\frac{1}{6}, \frac{1}{5}) \bigcup \ldots$
Length of required interval:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2$$

Required probability $= \frac{\text{length of required interval}}{\text{total length of interval}}$
$\therefore$ Required probability $= \frac{\ln 2}{1} = \ln 2$

Now, evaluating P(k) is simplified as N is the sum of 2k+1 integers which would be odd iff an odd number of those integers are odd and the rest are even, i.e., the following cases:
2k+1 odd, 0 even
2k-1 odd, 2 even
.
.
.
1 odd, 2k even
Treating this like a Bernoulli Trial, the required probability is given as:

$$P(k) = \binom{2k+1}{0} p^{2k+1} + \binom{2k+1}{2} p^{2k-1}(1-p)^2 + \ldots \binom{2k+1}{2k} p(1-p)^{2k}$$

$$= \sum_{i=0}^{k} \binom{2k+1}{2i} p^{2k+1-2i}(1-p)^{2i} \tag{1}$$

Where $p = \ln 2$, the probability of one term being odd.
To simplify (1), the following can be observed:

$$(x+y)^{2k+1} = \binom{2k+1}{0} x^{2k+1} + \binom{2k+1}{1} x^{2k} y^1 + \ldots \binom{2k+1}{2k} x^1 y^{2k} + \binom{2k+1}{2k+1} y^{2k+1}$$
$$(x-y)^{2k+1} = \binom{2k+1}{0} x^{2k+1} - \binom{2k+1}{1} x^{2k} y^1 + \ldots \binom{2k+1}{2k} x^1 y^{2k} - \binom{2k+1}{2k+1} y^{2k+1}$$

Adding the two:

$$\frac{(x+y)^{2k+1} + (x-y)^{2k+1}}{2} = \sum_{i=0}^{k} \binom{2k+1}{2i} x^{2k+1-2i} y^{2i}$$

Putting $x = p$ and $y = 1-p$ and using (1):

$$\frac{1 + (2p-1)^{2k+1}}{2} = P(k)$$
$$\therefore 2P(k) - 1 = (2p-1)^{2k+1} \tag{2}$$

Now, to evaluate the sum, it can be seen that is a geometric progression with first term $(2p-1)$ and common ratio $(2p-1)^2$ with $|2p-1|<1$.

$$\therefore \lim_{n\to\infty}\sum_{k=0}^{n}(2P(k)-1) = \frac{2p-1}{1-(2p-1)^2}$$

$$= \frac{2p-1}{4p-4p^2}$$

$$= \boxed{\frac{2\ln 2-1}{4\ln 2(1-\ln 2)}}$$

5.

First, if $c \geq 2$ then we claim no such f and g exist. Indeed, one simply takes $x=1$ to get $f(1)/g(1) \leq 0$, impossible.

For $c<2$, let $c = 2\cos\theta$, where $0<\theta<\pi$. We claim that f exists and has minimum degree equal to n, where n is defined as the smallest integer satisfying $\sin n\theta \leq 0$. In other words

$$n = \left\lceil \frac{\pi}{\arccos(c/2)} \right\rceil.$$

First we show that this is necessary. To see it, write explicitly

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-2}x^{n-2}$$

with each $a_i \geq 0$, and $a_{n-2} \neq 0$. Assume that n is such that $\sin(k\theta) \geq 0$ for $k = 1, \ldots, n-1$. Then, we have the following system of inequalities:

$$\begin{aligned}
a_1 &\geq 2\cos\theta \cdot a_0 \\
a_0 + a_2 &\geq 2\cos\theta \cdot a_1 \\
a_1 + a_3 &\geq 2\cos\theta \cdot a_2 \\
&\vdots \\
a_{n-5} + a_{n-3} &\geq 2\cos\theta \cdot a_{n-4} \\
a_{n-4} + a_{n-2} &\geq 2\cos\theta \cdot a_{n-3} \\
a_{n-3} &\geq 2\cos\theta \cdot a_{n-2}.
\end{aligned}$$

Now, multiply the first equation by $\sin\theta$, the second equation by $\sin 2\theta$, et cetera, up to $\sin((n-1)\theta)$. This choice of weights is selected since we have

$$\sin(k\theta) + \sin((k+2)\theta) = 2\sin((k+1)\theta)\cos\theta$$

so that summing the entire expression cancels nearly all terms and leaves only

$$\sin((n-2)\theta)\, a_{n-2} \geq \sin((n-1)\theta) \cdot 2\cos\theta \cdot a_{n-2}$$

and so by dividing by a_{n-2} and using the same identity gives us $\sin(n\theta) \leq 0$, as claimed.

This bound is best possible, because the example

$$a_k = \sin((k+1)\theta) \geq 0$$

makes all inequalities above sharp, hence giving a working pair (f, g).

6.

We prove that the condition $x^4+y^4+z^4+xyz=4$ implies

$$\sqrt{2-x} \geq \frac{y+z}{2}.$$

We first establish that $2-x \geq 0$. Indeed, AM-GM gives that

$$5 = x^4+y^4+(z^4+1)+xyz = \frac{3x^4}{4} + \left(\frac{x^4}{4}+y^4\right) + (z^4+1) + xyz$$
$$\geq \frac{3x^4}{4} + x^2y^2 + 2z^2 + xyz.$$

We evidently have that $x^2y^2+2z^2+xyz \geq 0$ because the quadratic form $a^2+ab+2b^2$ is positive definite, so $x^4 \leq \frac{20}{3} \implies x \leq 2$. Now, the desired statement is implied by its square, so it suffices to show that

$$2-x \geq \left(\frac{y+z}{2}\right)^2$$

Assume for contradiction the reverse inequality holds. This rearranges to

$$4x+y^2+2yz+z^2 > 8.$$

By AM-GM, we have $x^4+3 \geq 4x$ and $\frac{y^4+1}{2} \geq y^2$ which yields that

$$x^4 + \frac{y^4+z^4}{2} + 2yz + 4 > 8 \implies x^4 + \frac{y^4+z^4}{2} + 2yz > 4.$$

Subtracting the given condition $x^4+y^4+z^4+xyz=4$ now gives

$$-\frac{y^4+z^4}{2} + (2-x)yz > 0 \implies (2-x)yz > \frac{y^4+z^4}{2}.$$

Since $2-x$ and the right-hand side are positive, we have $yz \geq 0$. So, we have

$$\frac{y^4+z^4}{2yz} < 2-x < \left(\frac{y+z}{2}\right)^2 \implies 2y^4+2z^4 < yz(y+z)^2 = y^3z+2y^2z^2+yz^3.$$

This is clearly false by AM-GM, so we have a contradiction.

7.

Solution 1. Let t be a real number with $0 < t < \sqrt{2}$. Taking $x = t/\sqrt{2}$ and $x = \sqrt{(2-t^2)/2}$ in the given yields:

$$P(t) = P\left(t/\sqrt{2} + \sqrt{1 - t^2/2}\right) = P\left(\sqrt{2 - t^2}\right)$$

We now decompose P into its odd and even parts. Let $P(x) = Q(x^2) + x \cdot R(x^2)$. If R is not the zero polynomial, plugging in $x = \sqrt{2 - t^2}$ into the above and rearranging yields:

$$\begin{aligned}\sqrt{2-t^2} &= \frac{P\left(\sqrt{2-t^2}\right) - Q(2-t^2)}{R(2-t^2)} \\ &= \frac{P(t) - Q(2-t^2)}{R(2-t^2)}\end{aligned}$$

But this implies that we can express $\sqrt{2 - t^2}$ as a rational function in t for all $t \in (0, 1)$ with $R(t) \neq 0$, a contradiction. So $R(x) = 0$ for all x and $P(x) = Q(x^2)$.

We now note that, for all r with $0 < r < 1$, we have $Q(r) = Q(2 - r)$. Since this holds for infinitely many r, it must be a polynomial identity. So $Q(1 + x)$ is an even polynomial, and so we can let $Q(x) = A((x - 1)^2)$ for some polynomial A.

Retruning to the origional functional equation, we note that:

$$P(\sqrt{2}x) = Q(2x^2) = A(4x^4 - 4x^2 + 1)$$

And:

$$P(x + \sqrt{1 - x^2}) = Q(1 + 2x\sqrt{1 - x^2}) = A(4x^2 - 4x)$$

So for all $x \in (0, 1)$, $A(4x^2 - 4x^4) = A(1 - (4x^2 - 4x^4))$. Since there are infinitely many possible values of $4x^2 - 4x^4$, we must have that $A(x) = A(1 - x)$ is a polynomial identity. So $A(\frac{1}{2} + x)$ is an even polynomial, and so $A(x) = B((x - \frac{1}{2})^2)$ for some polynomial B.

Thus, we have that $P(x)$ must satisfy $P(x) = B(P_0(x))$ for some polynomial B, where $P_0(x) = [(x^2 - 1)^2 - \frac{1}{2}]^2$. It is easy to verify that P_0 is indeed a solution, so all polynomials in P_0 must be solutions as well. The proof is complete.